

Gram : Dnyanteerth
Phone (02462) 229300, 229242
Fax : (02462) 259461

**SWAMI RAMANAND TEERTH
MARATHWADA UNIVERSITY,**
"Dnyanteerth", Vishnupuri,
NANDED – 431 606 (M.S.)



तार - ज्ञानतीर्थ
दुरध्वनी (०२४६२) २२९३००, २२९२४२
फैक्स (०२४६२) २५९४६९

**स्वामी रामानंद तीर्थ
मराठवाडा विद्यापीठ,
"ज्ञानतीर्थ" विष्णुपुरी,
नांदेड - ४३९ ६०६ (म.रा.)**

APDS/UGC/Minor-Final R./2013-14/ 728

August 12, 2013

14

To
The Joint Secretary,
University Grants Commission,
Western Regional Office,
Ganesh Khind,
Pune - 411 007.

Sub: Submission of Utilization Certificate and final work done report
Ref: UGC letter F.23-1779/10(WRO), dt.14.10.2010

Sir,

Kindly find enclosed herewith Utilization Certificate & Audited Statement with Final work done report for Minor research project in the subject of Mathematics, entitled "Contraction and Contractive Single Valued Map in D-Metric Spaces for fixed Point" sanctioned to Dr. S. S. Biradar, Shri Hawgiswami Mahavidyalaya, Udgir, Dist. Latur submitted by Principal of the college.

Kindly accept the same and release the final installment as per UGC rules.

Yours faithfully,


Registrar

Encl: As above.

Copy to:

1. The Principal, Shri Hawgiswami Mahavidyalaya, Dist. Latur.
2. Dr. S. S. Biradar, Shri Hawgiswami Mahavidyalaya, Udgir, Dist. Latur

Estd. : 1972

सत्यं शिवं सुंदरम्

Principal : 254922

Off.: 256166

Resi.: 256174



Bharat Liberal Education Society's
SHRI HAVAGISWAMI COLLEGE, UDGIR

Pin 413 517

Dist. Latur (Maharashtra)

(Affiliated to the Swami Ramanand Teerth Marathwada University, Nanded)

ARTS * COMMERCE * SCIENCE

Ref. No. 235/2013

Date : 21/8/13

Dr. B. G. Velapurkar

M.A.B.Ed., Ph.D.

Principal

Ref. No.: MRP./2012-13/

Date: 18-02-2013

To

Joint Secretary,
University Grants Commission,
Western Regional Office,
Pune University Campus,
PUNE-7.

Through

The Director

Board of College & University Development
Swami Ramanand Teerth Marathwada University,
NaNDED.

Subject :- Submission of Final progress report, Utilization Certificate of Minor Research Project and Release of Final Installment...

Respected Sir/Madam,


With reference to the subject cited above, please find enclosed herewith the Progress Report and Utilization certificate of Minor Research Project of **Dr. Biradar S.S.** Dept. of Mathematics, Shri Hawgi Swami Mahavidyalaya, Udgir, Dist. Latur, Maharashtra. The project work is completed within the stipulated period.

For the project the sum of Rs. 75,510/- (Rs. Seventy Five Thousand Five Hundred Ten Only) are spent on prescribed heads, where as sum of Rs. 55,000/- (Rs. Fifty Five Thousand Only) was received from UGC as First Installment.

Kindly make arrangement to release the final installment of the Project.

Thanking you,

Yours Faithfully


PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur

- Encl: 1. Final Progress Report.
2. Audited Utilization Certificate.
3. Audited Statement of Expenditure with item wise details under non-recurring and recurring heads.
4. Assets and Accession Certificate.
5. Brief Final Project Report (summary).

14 OCT 2010

Phones: (020) 25691477.
25691178, 25696897
Fax: (020) 25691477
Web site: www.ugc.ac.in

File No: 23-1779/10 (WRO)

The Accounts Officer
University Grants Commission
Ganeshkhind, Pune-411007.

Subject: Financial assistance to college teachers for undertaking Minor Research Projects -
Release of first installment.

Sir,

The UGC on the recommendations of the Expert Committee has approved the Minor Research Project entitled "Contraction and Contractive Single Valued Map in D-Metric Spaces. For Fixed Point" in the subject- **Mathematics** to be undertaken by **Smt. Biradar S. S., Shri Havagiswami College, Udgir, Latur- 413517**. The financial assistance of the UGC would be limited to Rs. 75000/- (Rupees Seventy five thousand Only) for a period of two years. An amount of Rs. 55000/- (Rupees Forty five thousand Only) is presently being sanctioned as the first installment.

Non-Recurring Grant for Two years	Amount (Rs)	Recurring grant	1 st Year Amount	2 nd Year Amount
Books & Journals	10000	Contingency	10000	10000
Equipment	25000	Special Needs	0	0
		Travel/Field work	10000	10000
		Chemicals & Glassware	0	0
		Others	0	0
Total (Rs.)	35000		20000	20000

Total amount for the project: Rs. 75000/-

The grant is subject to the terms and conditions as mentioned below.

1. A Certificate of Acceptance of the conditions governing the research project should be sent immediately to this office.
2. The amount of the grant shall be drawn by the Accounts Officer (D.D.O), University Grants Commission on the grant-in-aid bill and shall be disbursed to and credited to the above-mentioned institute through Cheque/D.D.
3. The sanctioned amount is debatable to the major Head 5.3.3. and is valid for payment during the financial year 2010 -2011 only.
4. The grant is subject to adjustment on the basis of Utilization Certificate in prescribed proforma submitted by University/College/Institute.

NOTE:

1. Date of implementation will be the date of sanction of first installment.
2. The researcher is required to submit an Acceptance Certificate of the project in the enclosed format to the affiliating university, which would then be sent to UGC (WRO) in a bunch by the University.
3. Please send one copy of the project completion report to Director, (NET/NET) Central

prop Smt -
S.S. Biradar
15-11-10

SHRI HAVAGISWAMI COLLEGE JGIR
INWARD NO. 522
DATE 22-11-10

1822

Final Report

On

Minor Research Project Entitled

**Contraction and Contractive Single Valued Map in D-Metric Spaces for
Fixed Point**

Submitted to

University Grants Commission,

Western Regional Office,

Pune- 411 007

Submitted by

Dr. Biradar S.S.

Dept. of Mathematics

Shri Hawgiswami Mahavidyalaya ,Udgir

Tq. Udgir Dist. Latur

2012

UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.

Annual/Final Report of the work done on the Major/Minor Research Project.

1. Project report No. : **Annual**
2. UGC Reference No. : **F. 23-1779/10 dated 14 Oct. 2010,**
Dr. BIRADAR S. S.
3. Period of report : **From 14 Oct. 2010 to 13 Oct. 2011**
4. Title of research project: **“Contraction and Contractive Single Valued Map in D-Metric Spaces for Fixed Point.”**
5. (a) Name of the Principal Investigator : **Dr. BIRADAR S. S.**
(b) Dept. and University/College where work has progressed :
Dept. of Mathematics, Shri Havagiswami College, Udgir Dist.Latur.
6. Effective date of starting of the project: **14 Oct. 2010**
7. Grant approved and expenditure incurred during the period of the report:
 - a. Total amount approved Rs. **75000/-**
 - b. Total expenditure Rs. : **75000/-**
- c. Report of the work done: (Please attach a separate sheet): **Separate sheet attached**
 - i. Brief objective of the project: **Separate sheet attached.**
 - ii. Work done so far and results achieved and publications, if any, resulting from the work (Give details of the papers and names of the journals in which it has been published or accepted for publication : -----
 - iii. Has the progress been according to original plan of work and towards achieving the objective. if not, state reasons : **Yes**
 - iv. Please indicate the difficulties, if any, experienced in implementing the Project: **Nil**
 - v. If project has not been completed, please indicate the approximate time by which it is likely to be completed. A summary of the work done for the period (Annual basis) may please be sent to the Commission on a


separate sheet : Nil

vi. If the project has been completed, please enclose a summary of the findings of the study. Two bound copies of the final report of work done may also be sent to the Commission : **Summary of the findings enclosed**

vii. Any other information which would help in evaluation of work done on the project. At the completion of the project, the first report should indicate the output, such as (a) Manpower trained (b) Ph. D. awarded (c) Publication of results (d) other impact, if any : Nil



**SIGNATURE OF THE
PRINCIPAL INVESTIGATOR**



**PRINCIPAL
PRINCIPAL**
Shri Havagiswami College
Udgir-413517 Dist. Latur



ज्ञान - विज्ञानं विमुक्तये

Annexure -III

UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.

ANNUAL/FINAL REPORT OF THE WORK DONE ON THE MINOR RESEARCH PROJECT (REPORT TO BE SUBMITTED WITHIN 6 WEEKS AFTER COMPLETION OF EACH YEAR)

1. Project report No. : **Final**
2. UGC Reference No. : **F. 23-1779/10 dated 14 Oct. 2010,**
Dr. BIRADAR S. S.
3. Period of report : **From 14 Oct. 2010 To 14 Oct. 2012**
4. Title of research project: **“Contraction and Contractive Single Valued Map in D-Metric Spaces for Fixed Point.”**
5. (a) Name of the Principal Investigator : **Dr. BIRADAR S. S.**
(b) Dept. and University/College where work has progressed :
Dept. of Mathematics, Shri Havagiswami College, Udgir Dist.Latur.
6. Effective date of starting of the project: **14 Oct. 2010**
7. Grant approved and expenditure incurred during the period of the report:
 - a. Total amount approved Rs. **75000/-**
 - b. Total expenditure Rs. : **75000/-**
- c. Report of the work done: (Please attach a separate sheet): **Separate sheet attached**
 - i. Brief objective of the project: **Separate sheet attached.**
 - ii. Work done so far and results achieved and publications, if any, resulting from the work (Give details of the papers and names of the journals in which it has been published or accepted for publication : -----
 - iii. Has the progress been according to original plan of work and towards Achieving the objective. if not, state reasons : **Yes**

- iv) Please indicate the difficulties, if any, experienced in implementing the project:
No
- (v) If project has not been completed; please indicate the approximate time by which it is likely to be completed. A summary of the work done for the period (Annual basis) may please be sent to the Commission on a separate sheet: **NA**
- (vi) If the project has been completed, please enclose a summary of the findings of the study. Two bound copies of the final report of work done may also be sent to the Commission: **Summary of the findings enclosed**
- (vii) Any other information which would help in evaluation of work done on the project. At the completion of the project, the first report should indicate the output, such as (a) Manpower trained (b) Ph. D. awarded (c) Publication of results (d) other impact, if any: **NA**

Siradar.

**SIGNATURE OF THE PRINCIPAL
INVESTIGATOR**

[Handwritten Signature]
PRINCIPAL
REGISTRAR/PRINCIPAL
Shri Havagishwami College
Udgir-413517 Dist. Latur



**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

UTILIZATION CERTIFICATE

Certified that the grant of **Rs. 75000/-** (Rupees Seventy five Thousand only) received from the University Grants Commission under the scheme of support for Minor Research Project entitled “Contraction and Contractive Single Valued Map in D-Metric Spaces for Fixed Point.” Vide UGC letter No. 23-1779/10 dated 14 October 2010, P.I. Dr. Biradar S.S. has been fully utilized for the purpose for which it was sanctioned and in accordance with the terms and conditions laid down by the University Grants Commission.

Biradar

SIGNATURE OF THE
PRINCIPAL INVESTIGATOR

Biradar

PRINCIPAL

Shri Havagiswami College
Udgir-413517 Dist. Latur

Rameshwar

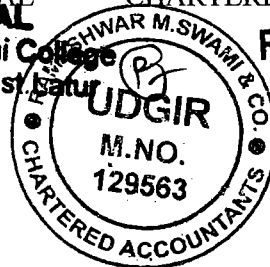
CHARTERED ACCOUNTANT

Rameshwar M. Swami & Co.

M.Com. A.C.A.

Chartered Accountants
“ Swami Chambers ”

Degloor Road UDGIR Dist. Latur
Mobile No. 9823272491





**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

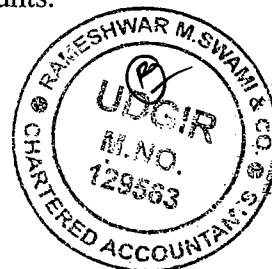
**STATEMENT OF EXPENDITURE IN RESPECT OF MAJOR/MINOR
RESEARCH PROJECT**

1. Name of Principal Investigator: **Dr. Biradar S.S.**
2. Deptt. of University/College: Department of Mathematics Shri Hawgiswmi Mahavidyalaya, Udgir Dist. Latur
3. UGC approval No. and Date: **F. 23-1779/10 dated 14 October 2010,**
4. Title of the Research Project: “Contraction and Contractive Single Valued Map in D-Metric Spaces for Fixed Point.”
5. Effective date of starting the project: 14 October 2010
6. (a) Period of Expenditure: From 14 October 2010 to 14 October 2012
- h. Details of Expenditure:

S. No.	Item	Amount Approved Rs.	Expenditure Incurred Rs.
i.	Books & Journals	10,000/-	10160/-
ii	Contingency	20,000/-	20090/-
iii.	Fieldwork/Travel	20,000/-	20150/-
iv	Equipment	25,000/-	25110/-
	Total	75,000/-	75510/-

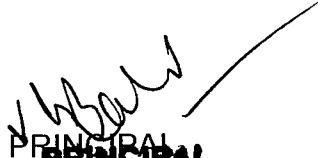
i. Staff: NA

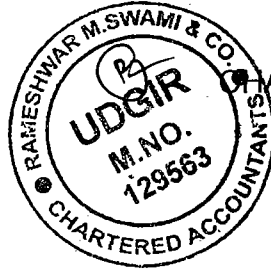
1. It is certified that the appointment(s) have been made in accordance with the terms and conditions laid down by the Commission.
2. It as a result of check or audit objective, some irregularly is noticed, later date, action will be taken to refund, adjust or regularize the objected amounts.




3. Payment @ revised rates shall be made with arrears on the availability of additional funds.

4. It is certified that the grant of **Rs. 75,000/-** (Rupees Seventy Five Thousand only) received from the University Grants Commission under the scheme of support for Minor Research Project entitled "Contraction and Contractive Single Valued Map in D-Metric Spaces for Fixed Point." Vide UGC letter No. 23-1779/10 dated 14 October 2010, P.I. Dr. Biradar S.S. has been fully utilized for the purpose for which it was sanctioned and in accordance with the terms and conditions laid down by the University Grants Commission.


PRINCIPAL
PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur




CHARTERED ACCOUNTANT
Rameshwar M. Swami & Co.
M.Com. A.C.A.
Chartered Accountants
"Swami Chambers"
Degloor Road UDGIR Dist.Latur
Mobile No. 9823272491



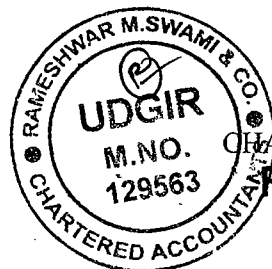
**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

**STATEMENT OF EXPENDITURE INCURRED ON TRAVEL & FIELD WORK
(Amount Approved Rs.20, 000/-)**

Name of the Principal Investigator: Dr. Biradar S.S.

Name of the place visited	Duration of the visit		Mode of journey	Expenditure incurred Rs. (TA+DA+LCA)
	Date	From – To		
Nanded	01/12/2010	Udgir to Nanded	By Car	2500/-
Hyderabad	4/5/2011	Udgir to Hyderabad	By Car	4050/-
Gulberga University	14/6/2011	Udgir to Gulberga	By Car	2750/-
Pune	21/10/2011	Udgir to Pune	By Car	3760/-
Aurangabad	2/1/2012	Udgir to Auragabad	By Car	2350/-
Banglore	9/5/2012	Udgir to Banglore	By Train	4740/-
Total				20150/-

V. B. W.
PRINCIPAL
PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist. Latur



S. Rameshwar
CHARTERED ACCOUNTANT
Rameshwar M. Swami & Co.
M.Com. A.C.A.
Chartered Accountants
"Swami Chambers"
Degloor Road UDGIR Dist. Latur
Mobile No. 9823272491



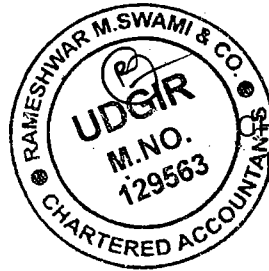
**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

**STATEMENT OF EXPENDITURE INCURRED ON BOOKS AND JOURNALS
(Amount Approved Rs.10, 000/-)**

Name of the Principal Investigator: Dr. Biradar S.S.

Particulars	Expenditure incurred Rs.
Krishna Book Agency, Udgir	5180/-
Krishna Book Agency, Udgir	4980/-
Total	10160/-

S.S. Biradar
PRINCIPAL
PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur



Rameshwar
CHARTERED ACCOUNTANT
Rameshwar M. Swami & Co.
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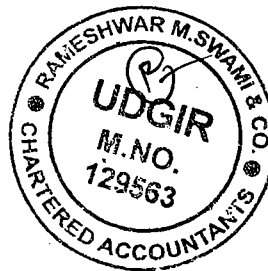
**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

**STATEMENT OF EXPENDITURE INCURRED ON CONTINGENCY
(Amount Approved Rs.20, 000/-)**

Name of the Principal Investigator: Dr. Biradar S.S.

Particulars	Expenditure incurred Rs.
Photocopy of Literature	4250/-
Photography	2500/-
Photo printing	1350/-
Computer typing	2000/-
Xerox	2100/-
Project binding	2500/-
Graphics software	5390/-
Total	20090/-

Dr. Biradar S.S.
PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur



Rameshwar
CHARTERED ACCOUNTANT
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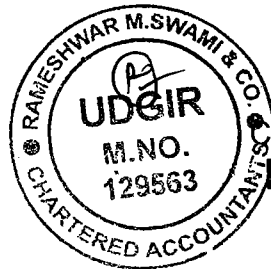
**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

**STATEMENT OF EXPENDITURE INCURRED ON EQUIPMENT
(Amount Approved Rs.25, 000/-)**

Name of the Principal Investigator: Dr. Biradar S.S.

Date	Particulars	Expenditure incurred Rs.
15/12/2010	LG Desktop Computer with Inter i3 Processor.	25110/-
	Total	25110/-

v. w. w.
PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur



R. Swamieshwar
CHARTERED ACCOUNTANT
Rameshwar M. Swami & Co.
M.Com. A.C.A.
Chartered Accountants
"Swami Chambers"
Degloor Road UDGIR Dist.Latur
Mobile No. 9823272491



ज्ञान-विज्ञान विमुक्तये

**UNIVERSITY GRANTS COMMISSION
BAHADUR SHAH ZAFAR MARG
NEW DELHI – 110 002.**

**PROFORMA FOR SUBMISSION OF INFORMATION AT THE TIME OF
SENDING THE FINAL REPORT OF THE WORK DONE ON THE PROJECT**

1. NAME AND ADDRESS OF THE PRINCIPAL INVESTIGATOR: **Dr. Biradar S.S.**
2. NAME AND ADDRESS OF THE INSTITUTION: **Department of Mathematics
Shri Havgiswami Mahavidyalaya ,Udgir Dist Latur.**
3. UGC APPROVAL NO. AND DATE: **23-1779/10 dated 14 October 2010**
4. DATE OF IMPLEMENTATION: **14 October 2010**
5. TENURE OF THE PROJECT: **From 14 October 2010 to 14 October 2012**
6. TOTAL GRANT ALLOCATED: **75,000/-**
7. TOTAL GRANT RECEIVED: **55,000/-**
8. FINAL EXPENDITURE: **75,510/-**
9. TITLE OF THE PROJECT: **“Contraction and Contractive Single Valued Map in
D-Metric Spaces for Fixed Point.”**
10. OBJECTIVES OF THE PROJECT: **Separate Sheet is Attached**
11. WHETHER OBJECTIVES WERE ACHIEVED: **Yes**
(GIVE DETAILS)

12. ACHIEVEMENTS FROM THE PROJECT: Separate Sheet is Attached

13. SUMMARY OF THE FINDINGS (IN 500 WORDS): Summary attached

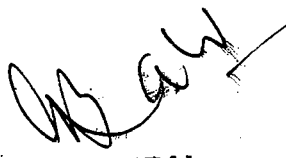
14. CONTRIBUTION TO THE SOCIETY (GIVE DETAILS): -----

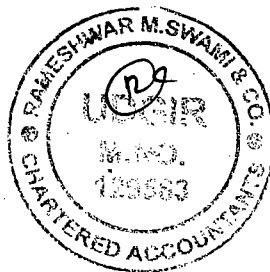
15. WHETHER ANY Ph.D. ENROLLED/PRODUCED

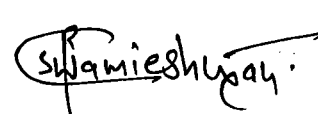
OUT OF THE PROJECT: No

16. NO. OF PUBLICATIONS OUT OF THE PROJECT (PLEASE ATTACH RE-

PRINTS) : A research paper is communicated for publication


PRINCIPAL
Shri Havagiswami College
Udgir-413517 Dist.Latur





RAMESHWAR M. SWAMI & CO.
CHARTERED ACCOUNTANTS
M.COM.FCA.
Bhure Complex, Near Sonwane
Petrol Pump, Degloor Road, udgir
Dist.latur Mobile No.9823272491

Assets certificate

It is certified that the following equipment have been handed over to the college.


- 1) Desktop computer


PRINCIPAL
Shri Hanagiswami College
Udgir-413517 Dist.Latur

Accession certificate

It is certified that the books purchased from MRP grant are handed over to the college central/departmental library.

Their accession number is from M-1 to M-6.


- **PRINCIPAL**
Shri Hanumanth College
Udgir-413517 Dist. Latur

DECLARATION

I hereby declare that the Minor Research Project entitled "**Contraction and Contractive Single Valued Map in D-Metric Spaces For Fixed Point.**" sanctioned by UGC is original work carried out by me in the Dept. of Mathematics, Shri Hawgi Swami Mahavidyalaya, Udgir, Dist. Latur, Maharashtra during 14/10/2010 to 14/10/2012.


Dr. Biradar S.S.

Dept. of Mathematics
Shri Hawgi Swami Mahavidyalaya,
Udgir, Dist. Latur, Maharashtra

Contraction and contractive single valued map in D-metric spaces for fixed point

Introduction:

This deals with the study of common fixed point theorems for two non-coming mappings S and T on a Hilbert space H into itself satisfying a generalized contraction condition of the form.

$$\begin{aligned} \|Sx - Ty\|^2 &\leq a(\|x - Sx\|^2 + \|y - Ty\|^2) \\ &+ b \max \{ \|x - y\|^2, \|x - Sx\|^2, \|y - Ty\|^2, \\ &\frac{1}{2} (\|x - Ty\|^2 + \|y - Sx\|^2) \} \quad (1) \end{aligned}$$

For all $x, y \in H$, where a and b are nonnegative real numbers such that

$$a + b < \frac{1}{2}$$

Fixed & common fixed point theorems for contractive & strict contractive single & set valued mappings are investigated by many authors.

In (1974) Ciric proved some non-unique fixed point theorem for orbitally continuous self-mappings on a T -orbitally complete metric space M which satisfy a condition of the type.

$$\begin{aligned} \min \{ d(Tx, Ty), d(x, Tx), d(y, Ty) \} \\ - \min \{ d(x, Ty), d(y, Tx) \} \leq q d(x, y) \quad (1.1) \end{aligned}$$

$$\min \left\{ (d(Tx, Ty))^2, d(x, y)d(Tx, Tx), (d(y, Ty))^2 \right\} \\ - \min \{ d(x, Ty)d(y, Ty), d(x, Ty)d(y, Tx) \} \leq q d(x, Tx)d(y, Ty) \quad (1.2)$$

for all $x, y \in M$ and for some $q \in (0, 1)$.

The purpose of this project is to improve upon the above results and establish some results of the maps with a nonunique fixed point. We give our main results in the following section.

In 1986 G. Jungck introduced the notion of compatible maps to the setting of single mappings in order to generalize the concept of commutativity & weak commutativity.

After words, the same other weakens above notion by introducing the notion of weak compatibility & recently with B.E. Rhoades, he extended the above notion to the setting of single & set valued maps.

In 2002 M. Aamri & D.E. Moutawakil defined a property (E.A) for self maps which contained the class of non compatible maps.

More recently T. Kamran extended the property (E.A.) for a hybrid pair of single & multivalued maps.

The aim of this paper is to give some new common fixed point theorems for single & set-valued maps under strict contractive conditions. These result unify improve extend & generalize the result by utilizing the property (E.A.).

We begin by stating some known definitions.

Preliminaries:

Let X be a metric space with metric d we denote by $(CB(X))$ the class of all non empty bounded closed subsets of X . we define the functions $\delta(A,B)$ & $D(A,B)$ as follows.

$$D(A,B) = \inf \{d(a,b) : a \in A, b \in B\}$$

$$\delta(A,B) = \sup \{d(a,b) : a \in A, b \in B\}$$

For all A, B in $(CB(X))$: if A contains a single point a , we write.

$$\delta(A,B) = \delta(a, B)$$

Also,

If B contains a single point b , it yields

$$\delta(A,B) = d(a, b)$$

the definition of the function $\delta(A,B)$ yields the following.

$$\delta(A,B) = \delta(B,A)$$

$$\delta(A,A) = \text{diam } A$$

For all A, B, C in $CB(X)$

Definition:

Maps $F: X \rightarrow X$ & $T: X \rightarrow CB(X)$ are weakly compatible if they compute at their coincidence points.

i.e. if $f(x) = Tfx$, wherever $fx \in Tx$

Definition:

Maps $F : X \rightarrow X$ & $g : X \rightarrow X$ are said to be satisfy the property (E.A.) if \exists a seqⁿ $\{x_n\}$ in x such that.

$$\lim_{n \rightarrow \infty} f x_n = \lim_{n \rightarrow \infty} g x_n = t \in x$$

Definition:

Maps $F : X \rightarrow X$ & $T, X \rightarrow CB(X)$ are said to be satisfy the property (E.A.) if \exists a seqⁿ $\{x_n\}$ in X some $t \in X$ &

$\Lambda \in CB(X)$ such that,

$$\lim_{n \rightarrow \infty} f x_n = t \in x = \lim_{n \rightarrow \infty} T x_n =$$

Theorem 2.1:

Let (x,d) be a metric let $F, G, X \rightarrow CB(X)$ & $I, J: X \rightarrow X$ be set & single valued mappings resp. satisfying the conditions.

- 1) $UF(x) \leq J(x)$ & $UG(x) \leq I(x)$
- 2) The inequality

$$\delta(Fx, Gy) < \alpha \max \{d(Ix, Jy) \delta(Ix, Fx) \delta(Jy, Gy)\} + (1-\alpha) [aD(Ix, Gy) + bD(Jy, Fx)]$$

For all $x, y \in X$, where.

- 3) $0 \leq \alpha < 1, 0 \leq a \leq \frac{1}{2}, 0 \leq b \leq \frac{1}{2}$.

Holds whenever the right hand side of (2) is positive.

If either:

4) F, I are weakly compatible & satisfy property (E.A.), G, J are weakly compatible & $UF(x)$ (resp. $J(x)$) is closed, J are weakly compatible & satisfy property (E.A), F, I are weakly compatible & $UG(x)$ (resp. $J(x)$) is closed.

Then there is a unique common fixed point

Z in X such that.

$$Fz = Gz = \{z\} = \{Iz\} = \{Jz\} \quad \rightarrow \quad (1)$$

Proof:

Suppose that F & I satisfy property (E.A.) then there exists a sequence $\{x_n\}$ in $x, t \in x$ & $A \in CB(X)$ such that

$$\lim_{n \rightarrow \infty} Ix_n = t \in A = \lim_{n \rightarrow \infty} Fx_n$$

Since $UF(x)$ is closed & $UF(X) \leq J(X)$, then there exists a point u in X such that

$$Ju = t$$

The inequality (2) gives.

$$\begin{aligned} \delta(Fx_n, Gu) &< \alpha \max \{d(Ix_n, Ju), \delta(Ix_n, Fx_n)\delta(Ju, Gu)\} \\ &+ (1-\alpha)[aD(Ix_n, Gu) + bD(Ju, Fx_n)] \end{aligned}$$

Taking the limit as $n \rightarrow \infty$ yields

$$\begin{aligned} \delta(Ju, Gu) &\leq \alpha \max \{0, 0, \delta(Ju, Gu)\} + (1-\alpha)aD(Ju, Gu) \\ &= \alpha\delta(Ju, Gu) + (1-\alpha)aD(Ju, Gu) \\ &\leq [\alpha + (1-\alpha)a] \delta(Ju, Gu) \end{aligned}$$

It is obvious that $[\alpha + (1-\alpha)a] < 1$,

Then the above contradiction demand that

$$Gu = \{Ju\}$$

Since G & J are weakly compatible.

$Gu = \{Ju\}$ implies that

$$GJu = JGu \text{ \& hence } GG_u = GJu = JGu = \{JJ_u\} \rightarrow (2)$$

Again by inequality (2) we have

$$\begin{aligned} \delta(Ix_n, GG_u) &< \alpha \max \{d(Ix_n, JGu) \delta(Ix_n, Fx_n) \delta(JGu, GG_u)\} \\ &+ (1-\alpha) [aD(Ix_n, GG_u) + bD(JGu, Fx_n)] \end{aligned}$$

By letting n tends to infinity we obtain,

$$\begin{aligned} \delta(Ju, GG_u) &\leq \alpha \max \{d(Ju, GG_u) 0, 0\} + (1-\alpha)(a+b)D(Ju, GG_u) \\ &= \alpha d(Ju, GG_u) + (1-\alpha)(a+b)D(Ju, GG_u) \\ [\alpha + (1-\alpha)(a+b)] &< 1, \end{aligned}$$

Then we have

$$GG_u = \{Ju\}$$

$$\text{Hence } \{Ju\} = GG_u = JGu$$

i.e. $Gu = GG_u = JGu$ & Gu is a common fixed point of G & J.

since $UG(X) \leq I(X)$

then there a point $V \in$ such that

$$\{Iv\} Gu$$

Moreover the use of (2) gives.

$$\begin{aligned}
\delta(Fu, Gu) &< \alpha \max \{d(Iv, Ju), \delta(Iv, Fv), \delta(Ju, Gu)\} \\
&\quad + (1-\alpha) [aD(Iv, Gu) + bD(Ju, Fv)] \\
&= \alpha \max \{0, \delta(Iv, Fv), 0\} + (1-\alpha) b.D.(Ju, Fv) \\
&= \alpha \delta(Gu, Fv) + (1-\alpha) b.D(Gu, Fv) \\
&\leq [\alpha E(1-\alpha)b] \delta(Gu, Fv)
\end{aligned}$$

It is easy to see that,

$$[\alpha + (1-\alpha)b] < 1 \text{ \&}$$

Therefore $Fv = Gu + \{Iv\}$

Since $Fc = \{Iv\}$ by the weak compatibility of

F & I, we get

$FIv = IFv$ & hence

$$FFv = FIv = IFv = \{IIv\} \quad \rightarrow \quad (3)$$

Moreover by (2) we can estimate.

$$\begin{aligned}
\delta(FFu, Gu) &< \alpha \{d(IFu, Ju), \delta(IFv, FFv), \delta(Ju, Gu)\} \\
&\quad + (1-\alpha) [aD(IFv, Gu) + bD(Ju, FFv)] \\
&= \alpha \max \{d(IFv, Ju), 0, 0\} + (1-\alpha)(a+b)D(IFv, Gu) \\
&= \alpha d(FFv, Gu) + (1-\alpha)(a+b)D(FFv, Gu) \\
&\leq [\alpha + (1-\alpha)(a+b)] \delta(FFv, Gu) < \delta(FFv, Gu)
\end{aligned}$$

Which is a contradiction.

Thus $FFv = Gu$

i.e. $FGu = Gu = IGu$ & Gu is also a common fixed point of F & I

let $z = Gu$, then

$$Fz = Gz = \{z\} = \{Iz\} = \{Iz\} \rightarrow (4)$$

Similarly, one can obtain this conclusion by using, (u) in lieu of (4)

Finally, we prove that z is unique.

Let z' be another common fixed point of the maps I, J, F & G such that

$$z' \neq z.$$

Then by estimation (2) we may get,

$$\begin{aligned} d(z, z') &= \delta(Fz, Gz') < \alpha \max \{d(Iz, Jz'), \delta(Iz, Fz)\} \\ &\quad \delta(Jz', Gz') + (1-\alpha) [aD(Iz, Gz) + bD(Jz, Fz)] \\ &= \alpha \max \{d(z, z'), 0, 0\} + (1-\alpha)(a+b)D(z, z') \\ &= \alpha d(z, z') + (1-\alpha)(a+b)D(z, z') \\ &\leq [\alpha + (1-\alpha)(a+b)]d(z, z') < d(z, z') \end{aligned}$$

This contradiction,

Implies that $z' = z$.

Hence z is the unique common fixed point of I, J, F & G .

If we let in above theorem $F = G$ & $I = J$ then we get the following result.

Corollary :

Let $I : X \rightarrow X$ be a self map of a metric space (x, d) & $F : X \rightarrow CB(X)$ a set valued map.

Assumes that F & I satisfy the conditions,

- i) F & I satisfy property (E.A.)
- ii) $U F(X) \leq I(X)$
- iii) The inequality,

$$\delta(Fx, Fy) < \alpha \max \{d(Ix, Iy), \delta(Ix, Fx), \delta(Iy, Fy)\} \\ + (1-\alpha)[aD(Ix, Fy) + bD(Iy, Fx)]$$

for all $x, y \in X$, where $0 \leq \alpha \leq 1$
 $0 \leq a \leq \frac{1}{2}, 0 \leq b < \frac{1}{2}$

Whenever the right hand side of the inequality (iii) is positive. If FD & I are weakly compatible & $UF(x)$ resp. $I(x)$ is closed. Then F & I have a unique common fixed point Z in X .

For three maps we have the following results.

Theorem :

Let $F, G : X \rightarrow CB(X)$ be set valued mappings & $I, J : X \rightarrow X$ be self mappings on the metric space X .

If inequality (2) holds For all $x, y \in X$ then.

$$(FI \cap FJ) \cap FF = (FI \cap FJ) \cap FG \quad \rightarrow \quad (5)$$

Proof:

Let $z \in (FI \cap FJ) \cap FF$

Then estimation (2) gives.

$$\delta(Fz, Gz) < \alpha \max \{d(Iz, Jz), \delta(Iz, Fz), \delta(Jz, Gz)\} \\ + (1-\alpha)[aD(Iz, Gz) + bD(Jz, Fz)]$$

therefore,

$$\delta(Z, Gz) < \max \{0, \delta(z, Gz)\} + (1-\alpha)aD(Z, Gz) \\ = \alpha\delta(Z, Gz) + (1-\alpha)aD(Z, Gz) \\ \leq [\alpha + (1-\alpha)a]\delta(Z, Gz) < \delta(Z, Gz)$$

This contradiction implies that,

$Gz = \{z\}$ thus

$$(F_1 \cap F_J) \cap F_F < (F_1 \cap F_J) \cap F_G \rightarrow (6)$$

Similarly,

$$(F_1 \cap F_J) \cap F_G < (F_1 \cap F_J) \cap F_F \rightarrow (7)$$

Theorem:

Let $I, J : X \rightarrow X$ be a self mappings &

$F : X \rightarrow CB(A), i \in N^* = \{1, 2, \dots\}$

Set valued maps such that,

i) $UF_1(x) \leq J(x) \& UF_2(x) \leq I(x)$

ii) The inequality,

$$\delta(f_{ix}, f_{i+1}y) < \alpha \max \{d(Ix, Jy), \delta(Ix, fix), \delta(Jy, F_{i+1}y)\} \\ + (1-\alpha)[aD(Ix, F_{i+1}y) + bD(Jy, fix)]$$

Holds for all $x, y \in X$, where $i \in N^*$

Where $0 \leq \alpha < 1, 0 \leq a < \frac{1}{2}, 0 \leq b < \frac{1}{2}$.

Whenever the right hand side of (ii) is positive further if either.

iii) F_1, I are weakly compatible satisfying property (E.A.) ; F_2, J are weakly compatible $UF_1(x)$ (resp. $J(x)$) is closed or

iv) F_2, J are weakly compatible satisfying property (E.A.) F, I are weakly compatible & $UF_2(x)$ (resp. $I(x)$) is closed.

Then there exists a unique common fixed point $Z \in X$ such that

$$FiZ = \{Iz\} = \{Jz\} = \{Z\} \quad \text{where } i \in N^* \rightarrow (8)$$

Proof:

The proof of this theorem follows from above two theorems (4) & (5).

Picard's Theorems:

Theorem :

Let F be a continuous real – valued function defined on $|x| \leq a,$

$|y| \leq b$ such that

i) $F(0,0) = 0$

ii) $|F(x_1, y_1) - F(x_2, y_2)| \leq k (|x_1 - x_2| + |y_1 - y_2|)$

Where k is a fixed no. in $(0,1)$

Then the equation $y = (x + F(x,y))$ has a unique solution $y = h(x)$ with $h(0) = 0$ & h defined in

$$|x| \leq S < \min \left\{ a, \frac{1-k}{|c|+k}, b \right\}$$

Furthermore,

$$|h(x_2) - h(x_1)| \leq \frac{|c| + k}{1 - k} |x_2 - x_1|$$

Proof:

Let X be the family of all continuous functions g defined on $(-5, 5)$ such that,

$$g(0) = 0 \text{ \& } |g(x)| \leq b$$

with the usual sup metric d_X is a complete metric space.

We define $T : X \rightarrow X$ by

$$Tg(x) = (x + F(x, g(x)))$$

Then T is well defined since,

i) $Tg(0) = F(0, 0) = 0$

ii) $T(g)$ is continuous.

iii)

$$\begin{aligned} |T(g)(x)| &\leq |cx| + |f(x, g(x))| \\ &\leq |cx| + k(s + b) \\ &< b, \quad \text{by def}^n \text{ of } S \end{aligned}$$

moreover,

$$|T(g_1)(x) - T(g_2)(x)| = |F(x, g_1)(x) - F(x, g_2)(x)|$$

this means that,

$$d(T(g_1), T(g_2)) \leq kd(g_1, g_2) \text{ by}$$

Banach, fixed point theorem T has a unique fixed point $h(x) \in X$

Hence the proof.

Now we prove two existence theorems for integral equations by using theorem.

Theorem :

Let the function $k(x,y)$ be defined & measurable in the square $A = \{(x,y) \mid a \leq x \leq b, a \leq y \leq b\}$

Further.

$$\int_a^b \int_a^b |k(x,y)|^2 dx dy < \infty$$

& $g(x) \in L_2(a,b)$ then the integral equation

$$f(x) = g(x) + \mu \int_a^b k(x,y) F(y) dy \rightarrow (1)$$

Possesses a unique solution $F(x) \in L_2(a,b)$

For every sufficiently small value of the parameter μ .

Proof:

For applying theorem 1.2 let $X = L_2$ & consider the mapping T

$$T : L_2(a,b) \rightarrow L_2(a,b)$$

$$Tf = h.$$

$$\text{When } h(x) = g(x) + \mu \int_a^b k(x,y) \cdot f(y) dy$$

This definition is valid for each $f \in L_2(a,b)$ & $h \in L_2(a,b)$

This can be seen as follows,

Since $g \in L_2(a,b)$ & λ is scalar, it is sufficient to show that,

$$\Psi(x) = \int_a^b k(x,y)f(y)dy \in L_2(a,b)$$

By the Cauchy – Schwarz inequality.

$$\begin{aligned} \left| \int_a^b k(x,y)f(y)dy \right| &\leq \int_a^b |k(x,y)f(y)| dy \\ &\leq \left(\int_a^b |k(x,y)|^2 dy \right)^{1/2} \left(\int_a^b |f(y)|^2 dy \right) \end{aligned}$$

therefore,

$$\begin{aligned} |\Psi(x)|^2 &= \left(\int_a^b k(x,y)f(y)dy \right)^2 \\ &\leq \left(\int_a^b |k(x,y)|^2 dy \right) \left(\int_a^b |f(y)|^2 dy \right) \\ \text{or } \int_a^b |\Psi(x)|^2 dx &\leq \int_a^b \left(\int_a^b |k(x,y)|^2 dy \right) dx \left(\int_a^b |f(y)|^2 dy \right) dx \end{aligned}$$

By the hypothesis,

$$\begin{aligned} \int_a^b \int_a^b |k(x,y)|^2 dx dy &< \infty \\ \&\int_a^b \left(\int_a^b |f(y)|^2 dy \right) dx < \infty \end{aligned}$$

$$\text{thus } \Psi(x) = \int_a^b k(x,y)f(y)dy \in L_2(a,b)$$

We know that $L_2(a,b)$ is a Banach space with norm

$$\|f\| = \left(\int_a^b |f(y)|^2 dy \right)^{1/2}$$

We now show that T is a contraction mapping

$$\|Tf - Tf_1\| = \|h - h_1\|$$

Where $h_1(x) = g_1(x) + \mu \int_a^b k(x,y)f(y)dy$

$$\begin{aligned} \|h - h_1\| &= \left\| \mu \int_a^b k(x,y)[f(y) - f_1(y)]dy \right\| \\ &= |\mu| \left(\int_a^b \left| \int_a^b k(x,y)(f(y) - f_1(y))dy \right|^2 dx \right)^{1/2} \\ &= |\mu| \left(\int_a^b \int_a^b |k(x,y)|^2 dx dy \right)^{1/2} \left(\int_a^b |f(y) - f_1(y)|^2 dy \right)^{1/2} \end{aligned}$$

By using Cauchy – Schwartz – Bunyakowshi inequality

Hence,

$$\|Tf - Tf_1\| \leq |\mu| \left(\int_a^b \int_a^b |k(x,y)|^2 dx dy \right)^{1/2} \|f - f_1\|$$

By definition of the norm in L_2 we have,

$$\|f - f_1\| = \left(\int_a^b |f(y) - f_1(y)|^2 dy \right)^{1/2}$$

$$\text{if } |\mu| < \frac{1}{\left(\int_a^b \int_a^b |k(x,y)|^2 dx dy \right)^{1/2}}$$

then, $\|Tf - Tf_1\| \leq k \|f - f_1\|$ where

$$0 \leq k \leq |\mu| \left(\int_a^b \int_a^b |k(x,y)|^2 dx dy \right)^{1/2} < 1$$

Thus T is a contraction,

& by theorems (2)

T has a unique fixed point

i.e. there exists $f^* \in L_2(a,b)$ such that

$$Tf^* = f^*$$

This fixed point f^* is a unique solution of (1).

Theorem:

Let a vector valued function $f(t,x)$ be defined on $k = I \times B$ into R^n

$$x(\lambda) = \bar{x}$$

$$\& B = \left\{ x(t) \in R^n \mid \|x(t) - \bar{x}\| \leq b \right\}$$

Be continuous & satisfy a Lipschitz condition

$$\|f(t,x) - f(t,y)\| \leq k \|x - y\|, t \in I, x, y \in B$$

further,

$$\text{let } \|f(t,x)\| \leq m \quad \text{for } (t,x) \in k \&$$

$$\text{set } c = \min\left(\frac{b}{m}, a, \frac{1-\epsilon}{k}\right), \epsilon > 0$$

then for $t \in [\lambda, \lambda + c]$, the integral equation

$$x(t) = \bar{x} + \int_{\lambda}^t (f(s, x(s))) ds \quad \rightarrow \quad (i)$$

Has a unique solution,

Proof:

Let $X = C([\lambda, \lambda + c], R^n)$ where,

$$\|x\| = \sup_{\lambda \leq t \leq \lambda + c} \|x(t)\|, x \in X$$

Define $T : X \rightarrow X$ by the relation.

$$Tx = y,$$

$$\text{where } y = \bar{x} + \int_a^t f(s, x(s)) ds$$

Let y be the subset of X of those x such that

$$\|x(t) - \bar{x}\|_{R^n} \leq b$$

then T maps y into y since for $\Psi \in y$

$$\|T\Psi(t) - \bar{x}\|_{R^n} = \left\| \bar{x} + \int_a^t f[s, \Psi(s)] ds - \bar{x} \right\|$$

$$\begin{aligned} \|T\Psi(t) - \bar{x}\|_{R^n} &= \left\| \int_a^t f(s, \Psi(s)) ds \right\| \\ &\leq \int_a^t \|f(s, \Psi(s))\|_{R^n} ds \\ &\leq M(t-a) \\ &\leq M_1 \leq b \end{aligned}$$

By the hypothesis,

Y is closed subset of x

To prove this,

$$\text{Let } \psi_n \in y, \psi_n \rightarrow \psi$$

Then,

$$\begin{aligned} \|\Psi(t) - \bar{x}\| &= \|\Psi(t) - \Psi_n(t) + \Psi_n(t) - \bar{x}\| \\ &\leq \|\Psi(t) - \Psi_n(t)\| + \|\Psi_n(t) - \bar{x}\| \\ &\leq b. \quad [\|\Psi(t) - \Psi_n(t)\| \rightarrow 0 \text{ as } n \rightarrow \infty] \end{aligned}$$

thus, $\Psi \in y$ & y is closed,

Now, we show that T is a contradiction mapping.

For $x, x^1 \in X$, we have

$$\begin{aligned} \|(Tx)(t) - (Tx^1)(t)\|_{R^n} &= \left\| \int_{\lambda}^t [F(S, x(s)) - F(S, x^1(s))] ds \right\|_{R^n} \\ &\leq \int_{\lambda}^t \|F(S, x(s)) - F(S, x^1(s))\|_{R^n} ds \\ &\leq k_1 \int_{\lambda}^t \|x(s) - x^1(s)\|_{R^n} ds \\ &\quad \text{[By Lipschitz condition on F]} \\ &\leq k_1 (t - \lambda) \|x - x^1\|_X \end{aligned}$$

therefore,

$$\begin{aligned} \|Tx - Tx^1\|_X &\leq k_1 C \|x - x^1\|_X \\ &= (1 - \epsilon) \|x - x^1\|_X \end{aligned}$$

Since, $0 \leq 1 - \epsilon < 1$

T is a contraction mapping.

By theorem \exists a unique fixed point x^1 of T , which is a unique solution of equation (1).

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